ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT
Journal of Algebra 283 (2005) 63–79
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Properly stratified endomorphism algebras

Xueqing Chen, Vlastimil Dlab *

School of Mathematics and Statistics, Carleton University, Ottawa, ON K1S 5B6, Canada

Received 23 April 2003

Communicated by Leonard L. Scott, Jr.

Abstract

In analogy to a previous result of Dlab–Heath–Marko on quasi-hereditary algebras, the paper provides sufficient and necessary conditions for particular endomorphism algebras to be properly stratified.

Keywords: Properly stratified algebras; Quasi-hereditary algebras; Endomorphism algebras

1. Introduction

Properly stratified algebras are related to the Futorny–König–Mazorchuk category of Lie representations [6] in a similar way as quasi-hereditary algebras to the category $\mathcal{O}$ of Bernstein–Gelfand–Gelfand. Let us add that the class of properly stratified algebras is naturally defined in terms of recollement of the respective derived categories of bounded complexes [4]. The impact of Dlab–Heath–Marko paper [5] on the theory of stratified algebras (cf. the seminal memoirs of Cline–Parshall–Scott [2]) suggests that the present result may play a similar role in more general situations.

The role of the endomorphism algebras in the structure of the category $\mathcal{O}$ has been already amplified by Soergel in his paper [9]. There, the algebra for the principal block of the category $\mathcal{O}$ is constructed algebraically as an endomorphism ring of a commuta-

* Corresponding author.
E-mail addresses: cxueqing@math.carleton.ca (X. Chen), vdlab@math.carleton.ca (V. Dlab).

0021-8693/$ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jalgebra.2004.09.002

tive algebra. Moreover, Ginzburg has shown in [7] that, under very broad conditions, the (Yoneda) Ext-algebras associated to perverse sheaves may be viewed as endomorphism algebras of corresponding hypercohomology modules of the commutative cohomology ring of the underlying space. His work suggests that many, if not most, representation theory categories realizable in terms of perverse sheaves, should have Ext-algebras realizable in terms of endomorphism rings associated to commutative algebras. This argues for more in depth studies concentrating on the commutative case. The present paper, as the sequel to [5], should be seen in this light.

2. Statements

Throughout this paper, K denotes an algebraically closed field and R a finite dimensional commutative local self-injective K-algebra. Furthermore, A denotes the endomorphism algebra of a direct sum X of pairwise nonisomorphic local–colocal R-modules $X(i)$, $1 \leqslant i \leqslant n$, i.e., R-modules such that both $X(i)/\operatorname{rad} X(i)$ and $\operatorname{soc} X(i)$ are simple. Write, for each $1 \leqslant i \leqslant n$, $e_i = p_i m_i$, where $p_i : X \to X(i)$ and $m_i : X(i) \to X$ are the canonical projection and embedding, respectively. Thus

$$A_A = \bigoplus_{i=1}^{n} e_i A$$

is the right regular representation of the basic algebra A; moreover, $P(i) = e_i A$, and $S(i) = e_i A/\operatorname{rad} e_i A$ are the right indecomposable projective and simple A-modules, respectively.

Now, for each i, $1 \leqslant i \leqslant n$, there is a unique submodule $\underline{X}(i)$ of ${}_R R$ that is isomorphic to $X(i)$. Thus $\underline{X}(i)$ is a local ideal of R. Observe that, for a given homomorphism $f : X(i) \to X(j)$, the respective $\underline{f} : \underline{X}(i) \to \underline{X}(j)$ can be extended to a module homomorphism of R to R and is therefore given by a multiplication (by a fixed element). Consequently, $\underline{f}(\underline{X}(i)) \subseteq \underline{X}(i)$, and thus $f((X(i))$ is isomorphic to a submodule of $X(i)$. Hence, we get immediately the following proposition.

Proposition 2.1 (See [5]). *Let $X(i)$ and $X(j)$ be local–colocal R-modules. The following three statements are equivalent*:

(1) $\underline{X}(i) \subseteq \underline{X}(j) \subseteq R$;
(2) *there is a monomorphism from $X(i)$ to $X(j)$*;
(3) *there is an epimorphism from $X(j)$ to $X(i)$.*

Furthermore, as in [5], since $\operatorname{Hom}_K(X(i), K) \simeq X(i)$, the algebra A possesses an involution $* : A \to A$ such that $e_i^* = e_i$ for all $1 \leqslant i \leqslant n$. Hence, there is a duality functor $\mathfrak{D} : \text{mod-}A \to \text{mod-}A$ such that

$$\mathfrak{D}\big(S(i)\big) \simeq S(i) \quad \text{for all } 1 \leqslant i \leqslant n.$$

Let us recall, for the benefit of a reader, the basic definitions concerning the stratification of algebras by standard and properly standard modules.

Given $(A, \mathbf{e})$, i.e., a K-algebra A together with an ordered complete sequence of primitive orthogonal idempotents $\mathbf{e} = (e_1, e_2, \dots, e_n)$, the sequence of the right standard A-modules which is defined by

$$\Delta = \big(\Delta(1), \Delta(2), \dots, \Delta(n)\big),$$

where

$$\Delta(i) = e_i A / e_i \operatorname{rad} A(e_{i+1} + e_{i+2} + \cdots + e_n)A, \quad 1 \leqslant i \leqslant n.$$

Similarly, the sequence of the left standard A-modules

$$\Delta^o = \big(\Delta^o(1), \Delta^o(2), \dots, \Delta^o(n)\big)$$

is defined. Furthermore,

$$\bar{\Delta} = \big(\bar{\Delta}(1), \bar{\Delta}(2), \dots, \bar{\Delta}(n)\big),$$

where

$$\bar{\Delta}(i) = e_i A / e_i \operatorname{rad} A(e_i + e_{i+1} + \cdots + e_n)A, \quad 1 \leqslant i \leqslant n,$$

is the sequence of the right proper standard A-modules. Again

$$\bar{\Delta}^o = \big(\bar{\Delta}^o(1), \bar{\Delta}^o(2), \dots, \bar{\Delta}^o(n)\big)$$

is defined similarly.

In the sequel, we shall deal with the following classes of algebras.

Definition 2.2 (See [1–4]). An algebra $(A, \mathbf{e})$ is called

(i) quasi-hereditary, if $A_A \in \mathcal{F}(\Delta)$ and $\Delta = \bar{\Delta}$;
(ii) properly stratified, if $A_A \in \mathcal{F}(\bar{\Delta}) \cap \mathcal{F}(\Delta)$;
(iii) standardly stratified, if $A_A \in \mathcal{F}(\bar{\Delta})$.

Here, the symbol $M_A \in \mathcal{F}(\Omega)$ means the right A-module M is filtered by A-modules from a family Ω, i.e., there is a finite chain of submodules

$$M_A = M_1 \supset M_2 \supset \cdots \supset M_t \supset M_{t+1} \supset \cdots \supset M_{d+1} = 0$$

such that $M_t / M_{t+1} \in \Omega$ for all $1 \leqslant t \leqslant d$.

Trivially, every quasi-hereditary algebra is properly stratified and every properly stratified algebra is standardly stratified.

Let us remark that our concept of standardly stratified algebras is a particular case of the general concept of Cline–Parshall–Scott in [2], where the existence of a complete set of idempotents is not required.

In view of the main result of [4] (see also [8]), namely, that

$$A_A \in \mathcal{F}(\bar{\Delta}) \quad \text{if and only if} \quad {}_A A \in \mathcal{F}\big(\Delta^o\big),$$

we have the following proposition.

Proposition 2.3. *Let $(A, \mathbf{e})$ be a K-algebra with duality $\mathfrak{D}:\text{mod-}A \to \text{mod-}A$ such that $\mathfrak{D}(S) \simeq S$ for all simple A-modules. Then $A_A \in \mathcal{F}(\Delta)$ if and only if $A_A \in \mathcal{F}(\bar{\Delta})$. Consequently $(A, \mathbf{e})$ is standardly stratified if and only if it is properly stratified.*

Now we are ready to state the main theorem of this paper.

Theorem 2.4. *Let R be a commutative local self-injective algebra over an algebraically closed field K. Let $\mathcal{X} = (X(1) = R, X(2), \ldots, X(n))$ be a sequence of local–colocal R-modules reflecting inclusions of the corresponding local ideals, i.e., $\underline{X}(i) \subseteq \underline{X}(j)$ implies $j \leqslant i$. Let $X = \bigoplus_{i=1}^n X(i)$, $A = \mathrm{End}_R(X)$, and $\mathbf{e} = (e_1, e_2, \ldots, e_n)$ be the sequence of the canonical idempotents of A. Then $(A, \mathbf{e})$ is a properly stratified algebra if and only if*

(i) *$\underline{X}(i) \cap \underline{X}(j)$ is generated by suitable $\underline{X}(t)$ of $\mathcal{X}$ for any $1 \leqslant i, j \leqslant n$;*
(ii) *$\underline{X}(j) \cap (\sum_{t=j+1}^n \underline{X}(t)) = \sum_{t=j+1}^n (\underline{X}(j) \cap \underline{X}(t))$ for any $1 \leqslant j \leqslant n$.*

3. Proof of sufficiency

For each i, $1 \leqslant i \leqslant n$, since $\underline{X}(i)$ is a local ideal of R, there is some $\underline{x}_i \in R$ such that $\underline{X}(i) = R\underline{x}_i$. Since every R-homomorphism $\underline{f}:\underline{X}(i) \to \underline{X}(j)$ can be extended to an R-homomorphism of ${}_R R$, $\underline{f}$ is given by multiplication. Thus $(\underline{x}_i)\underline{f} = r\underline{x}_i$ for a suitable $r \in R$.

For $1 \leqslant i \leqslant n$, fix an isomorphism μ_i from $X(i)$ to $\underline{X}(i)$. Write $x_i = (\underline{x}_i)\mu_i^{-1}$; thus, $X(i) = Rx_i$. Given a homomorphism $f : X(i) \to X(j)$, we know that $\mu_i^{-1} f \mu_j \in \mathrm{Hom}_R(\underline{X}(i), \underline{X}(j))$. The correspondence mapping $f \to \mu_i^{-1} f \mu_j$ is an R-isomorphism between the R-modules $\mathrm{Hom}_R(X(i), X(j))$ and $\mathrm{Hom}_R(\underline{X}(i), \underline{X}(j))$. In what follows, for any $f \in \mathrm{Hom}_R(X(i), X(j))$, write $\underline{f} = \mu_i^{-1} f \mu_j$. Thus for $fg : X(i) \xrightarrow{f} X(j) \xrightarrow{g} X(k)$, clearly $\underline{fg} = \underline{f}\,\underline{g}$.

The following R-homomorphisms will be used throughout the paper: if $\underline{X}(j) \subseteq \underline{X}(k)$, denote by $\underline{m}_{jk} : \underline{X}(j) \hookrightarrow \underline{X}(k)$ the canonical inclusion map (i.e., $(\underline{x}_j)\underline{m}_{jk} = \underline{x}_j$), and by $\underline{p}_{kj} : \underline{X}(k) \to \underline{X}(j)$ the projection given by $(\underline{x}_k)\underline{p}_{kj} = \underline{x}_j$. Correspondingly, we have a monomorphism

$$m_{jk} = \mu_j \underline{m}_{jk} \mu_k^{-1} : X(j) \to X(k)$$

and an epimorphism

$$p_{kj} = \mu_k \underline{p}_{kj} \mu_j^{-1} : X(k) \to X(j).$$

Put, for any $1 \leqslant i \leqslant n$, $\Lambda_i = \{1 \leqslant t \leqslant n \mid \underline{X}(t) \subseteq \underline{X}(i)\}$.

In addition to the indecomposable idempotents e_i, $1 \leqslant i \leqslant n$, of the (basic) endomorphism algebra $(A, \mathbf{e})$, we will consider also the idempotents $\varepsilon_i = e_i + e_{i+1} + \cdots + e_n$, $1 \leqslant i \leqslant n$; for convenience, put $\varepsilon_{n+1} = 0$.

In this section, we are going to show that $A_A \in \mathcal{F}(\Delta)$ and thus, in view of Proposition 2.3, that $(A, \mathbf{e})$ is properly stratified.

The statement $A_A \in \mathcal{F}(\Delta)$ will follow immediately from the following sequence of lemmas, in which conditions (i) and (ii) of Theorem 2.4 are always assumed. Note that condition (i) means that $\underline{X}(i) \cap \underline{X}(j) = \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{X}(t)$ for any $1 \leqslant i, j \leqslant n$.

Lemma 3.1. *Every R-homomorphism $f : X(i) \to X(j)$ factors through the module $\bigoplus_{t \in \Lambda_i \cap \Lambda_j} X(t)$.*

Proof. We will find the factorization of the map $\underline{f} : \underline{X}(i) \to \underline{X}(j)$. Since $\underline{f}$ is induced by multiplication, there is an element $r \in R$ such that $(\underline{x}_i)\underline{f} = r\underline{x}_i$. We know $\operatorname{Im} \underline{f} \subseteq \underline{X}(i) \cap \underline{X}(j) = \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{X}(t)$, hence $r\underline{x}_i = \sum_{t \in \Lambda_i \cap \Lambda_j} r_t \underline{x}_i$ with $r_t \underline{x}_i \in \underline{X}(t)$ for $r_t \in R$. For any $t \in \Lambda_i \cap \Lambda_j$, let us define $\underline{f}_t : \underline{X}(i) \to \underline{X}(t)$ with $(\underline{x}_i)\underline{f}_t = r_t \underline{x}_i$, then it is easy to show that $\underline{f}_t$ is a well-defined R-homomorphism. Hence, $\underline{f} = \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{f}_t \underline{m}_{tj}$. Thus,

$$f = \mu_i \underline{f} \mu_j^{-1} = \sum_{t \in \Lambda_i \cap \Lambda_j} \mu_i \underline{f}_t \mu_t^{-1} \mu_t \underline{m}_{tj} \mu_j^{-1} = \sum_{t \in \Lambda_i \cap \Lambda_j} f_t m_{tj}.$$

So we can write $f = gh$, with

$$g = (f_t)_{t \in \Lambda_i \cap \Lambda_j} : X(i) \longrightarrow \bigoplus_{t \in \Lambda_i \cap \Lambda_j} X(t) \quad \text{and}$$

$$h = (m_{tj})^T_{t \in \Lambda_i \cap \Lambda_j} : \bigoplus_{t \in \Lambda_i \cap \Lambda_j} X(t) \to X(j),$$

as required. $\square$

Lemma 3.2. *If an R-homomorphism $f : X(i) \to X(j)$ with $\underline{X}(i) \subseteq \underline{X}(j)$ cannot be factored through $\bigoplus_{t=i+1}^n X(t)$, then for any $\underline{X}(j) \subseteq \underline{X}(k)$, $f m_{jk} : X(i) \xrightarrow{f} X(j) \xrightarrow{m_{jk}} X(k)$ cannot be factored through $\bigoplus_{t=i+1}^n X(t)$.*

Proof. Let

$$f m_{jk} = X(i) \xrightarrow{f} X(j) \xrightarrow{m_{jk}} X(k) = X(i) \xrightarrow{(\cdots g_t \cdots)} \bigoplus_{t=i+1}^{n} X(t) \xrightarrow{(\cdots h_t \cdots)^T} X(k).$$

Then, fm_{jk} can be factored through $\bigoplus_{s\in\Lambda_i\setminus\{i\}} X(s)$ such that

$$fm_{jk} = X(i) \xrightarrow{(\cdots g'_s \cdots)} \bigoplus_{s\in\Lambda_i\setminus\{i\}} X(s) \xrightarrow{(\cdots h'_s \cdots)^T} X(k).$$

This follows from the fact that each g_t can be factored through $\bigoplus_{s\in\Lambda_i\setminus\{i\}} X(s)$ by Lemma 3.1, and the fact that all such $X(s)$ satisfy $\underline{X}(s) \subseteq \underline{X}(k)$.

Since each h'_s can be written in the form $h'_s = h''_s m_{sk}$, where h''_s is an endomorphism of $X(s)$, we have

$$fm_{jk} = \sum_{s\in\Lambda_i\setminus\{i\}} g'_s h'_s = \sum_{s\in\Lambda_i\setminus\{i\}} g'_s h''_s m_{sj} m_{jk},$$

and thus,

$$f = \sum_{s\in\Lambda_i\setminus\{i\}} g'_s h''_s m_{sj},$$

a contradiction. □

Lemma 3.3. *If an R-homomorphism $f : X(j) \to X(k)$ with $\underline{X}(j) \subseteq \underline{X}(k)$ cannot be factored through $\bigoplus_{t=j+1}^n X(t)$, then for any $\underline{X}(j) \subseteq \underline{X}(i)$, $p_{ij} f : X(i) \xrightarrow{p_{ij}} X(j) \xrightarrow{f} X(k)$ also cannot be factored through $\bigoplus_{t=j+1}^n X(t)$.*

Proof. For any local module $X(k)$, we have $\underline{X}(k) \subseteq \underline{X}(1) = R$. So if

$$p_{ij} f m_{k1} : X(i) \xrightarrow{p_{ij}} X(j) \xrightarrow{f} X(k) \xrightarrow{m_{k1}} X(1)$$

cannot be factored through $\bigoplus_{t=j+1}^n X(t)$, then $p_{ij} f$ cannot be factored through $\bigoplus_{t=j+1}^n X(t)$.

If $p_{ij} f m_{k1}$ can be factored through $\bigoplus_{t=j+1}^n X(t)$, then there exists a morphism

$$g = (g_{j+1}, g_{j+2}, \ldots, g_n) : X(i) \longrightarrow \bigoplus_{t=j+1}^n X(t) \quad \text{and}$$

$$h = (h_{j+1}, h_{j+2}, \ldots, h_n)^T : \bigoplus_{t=j+1}^n X(t) \longrightarrow X(1)$$

such that $p_{ij} f m_{k1} = gh$. Since the corresponding homomorphisms $\underline{f}$, $\underline{g}_t$, and $\underline{h}_t$ are induced by multiplication, we have $(\underline{x}_j)\underline{f} = r\underline{x}_j$, $(\underline{x}_i)\underline{g}_t = r_t\underline{x}_i$, and $(\underline{x}_t)\underline{h}_t = {r_t}'\underline{x}_t$, for $j+1 \leqslant t \leqslant n$ with $r, r_t, {r_t}' \in R$. Since $\underline{p}_{ij}\underline{f}\underline{m}_{k1} = \underline{gh} = \sum_{t=j+1}^n \underline{g}_t\underline{h}_t$, we know

$$r\underline{x}_j = \sum_{t=j+1}^{n} {r_t}' r_t \underline{x}_i \in \underline{X}(j) \cap \left(\sum_{t=j+1}^{n} \underline{X}(t) \right).$$

By the condition (ii),

$$\underline{X}(j) \cap \left(\sum_{t=j+1}^{n} \underline{X}(t) \right) = \sum_{t=j+1}^{n} \big(\underline{X}(j) \cap \underline{X}(t) \big),$$

and thus, $r\underline{x}_j = \sum_{t=j+1}^{n} r_t'' \underline{x}_j$, with $r_t'' \underline{x}_j \in \underline{X}(j) \cap \underline{X}(t)$ for $j+1 \leqslant t \leqslant n$ and $r_t'' \in R$. Define $\underline{g}_t' : \underline{X}(j) \to \underline{X}(t)$ by $(\underline{x}_j)\underline{g}_t' = r_t'' \underline{x}_j$, hence $\underline{f}\underline{m}_{k1} = \sum_{t=j+1}^{n} \underline{g}_t' \underline{m}_{t1}$. Therefore $f m_{k1} = \sum_{t=j+1}^{n} g_t' m_{t1}$, consequently, $f m_{k1}$ can be factored through $\bigoplus_{t=j+1}^{n} X(t)$ and by Lemma 3.2, f can be factored through $\bigoplus_{t=j+1}^{n} X(t)$, a contradiction. The lemma follows. □

Lemma 3.4. *Let $1 \leqslant i < j \leqslant n$. If $\underline{X}(j) \nsubseteq \underline{X}(i)$, then $e_i A \varepsilon_j A / e_i A \varepsilon_{j+1} A = 0$. If $\underline{X}(j) \subseteq \underline{X}(i)$, then $e_i A \varepsilon_j A / e_i A \varepsilon_{j+1} A \simeq \Delta(j)$.*

Proof. If $\underline{X}(j) \nsubseteq \underline{X}(i)$, then, by Lemma 3.1, any homomorphism from $X(i)$ to $X(j)$ can be factored through $\bigoplus_{s \in \Lambda_i \cap \Lambda_j} X(s)$. But $\Lambda_i \cap \Lambda_j \subseteq \Lambda_j \setminus \{j\} \subseteq \{j+1, j+2, \ldots, n\}$, and therefore $e_i A e_j A \subseteq e_i A \varepsilon_{j+1} A$; thus, $e_i A \varepsilon_j A = e_i A \varepsilon_{j+1} A$.

If $\underline{X}(j) \subseteq \underline{X}(i)$, we know that $\underline{p_{ij}}$ is the surjective homomorphism from $\underline{X}(i)$ to $\underline{X}(j)$ which maps $\underline{x}_i$ to $\underline{x}_j = r\underline{x}_i$ for some $r \in R$. Given any homomorphism $\underline{h} : \underline{X}(i) \to \underline{X}(j)$, suppose $(\underline{x}_i)\underline{h} = r'\underline{x}_i$ for some $r' \in R$. Since $\operatorname{Im} \underline{h} \subseteq \underline{X}(j)$, there exists some $r_j \in R$ such that $(\underline{x}_i)\underline{h} = r'\underline{x}_i = r_j \underline{x}_j$. Hence we can write $\underline{h} = \underline{p_{ij} q}$ with $\underline{q} : \underline{X}(j) \to \underline{X}(j)$, $(\underline{x}_j)\underline{q} = r_j \underline{x}_j$. As a consequence, we have the corresponding homomorphism $h : X(i) \to X(j)$, $h = p_{ij} q$ with q an endomorphism of $X(j)$. Thus defining the element $\pi_{ij} = p_i p_{ij} m_j$ in A, it is easy to show that $e_j A \simeq e_i \pi_{ij} e_j A$, $e_j A \varepsilon_{j+1} A \simeq e_i \pi_{ij} e_j A \varepsilon_{j+1} A$, $e_i A e_j A = e_i \pi_{ij} e_j A \simeq e_j A$ and $\Delta(j) \simeq e_i \pi_{ij} e_j A / e_i \pi_{ij} e_j A \varepsilon_{j+1} A$.

We know that $e_i A \varepsilon_j A / e_i A \varepsilon_{j+1} A$ is isomorphic to a quotient of $P(j) = e_j A$, since

$$e_i A \varepsilon_j A / e_i A \varepsilon_{j+1} A \simeq e_i A e_j A / e_i A \varepsilon_{j+1} A \cap e_i A e_j A.$$

So to complete the proof, we need to show

$$e_i \pi_{ij} e_j A \varepsilon_{j+1} A = e_i A \varepsilon_{j+1} A \cap e_i A e_j A,$$

or equivalently

$$e_i \pi_{ij} e_j A \varepsilon_{j+1} A e_l \supseteq e_i A \varepsilon_{j+1} A e_l \cap e_i A e_j A e_l = e_i A \varepsilon_{j+1} A e_l \cap e_i \pi_{ij} e_j A e_l$$

for every $1 \leqslant l \leqslant n$.

For any $1 \leqslant l \leqslant n$, there are three possibilities for the relation between $\underline{X}(l)$ and $\underline{X}(j)$:

(1) If $\underline{X}(l) \subset \underline{X}(j)$, then $l > j$ and $X(l)$ is a direct summand of $\bigoplus_{t=j+1}^{n} X(t)$. It is trivial that $e_i\pi_{ij}e_jAe_l \subseteq e_i\pi_{ij}e_jA\varepsilon_{j+1}Ae_l$.
(2) If $\underline{X}(l)$ and $\underline{X}(j)$ are incomparable, then any homomorphism from $X(j)$ to $X(l)$ can be factored through $\bigoplus_{s\in\Lambda_j\cap\Lambda_l} X(s)$ by Lemma 3.1. But $\Lambda_j \cap \Lambda_l \subseteq \Lambda_j \setminus \{j\} \subseteq \{j+1, j+2, \ldots, n\}$. So $e_i\pi_{ij}e_jAe_l \subseteq e_i\pi_{ij}e_jA\varepsilon_{j+1}Ae_l$.
(3) If $\underline{X}(j) \subseteq \underline{X}(l)$, then the elements of $e_iA\varepsilon_{j+1}Ae_l$ are sums of homomorphism a of the form $a = p_ighm_l$ where $g : X(i) \to \bigoplus_{t=j+1}^{n} X(t)$ and $h : \bigoplus_{t=j+1}^{n} X(t) \to X(l)$. Moreover, if $a \in e_i\pi_{ij}e_jAe_l$, then $a = p_ip_{ij}fm_l$ for some homomorphism $f : X(j) \to X(l)$. Thus, $gh = p_{ij}f$ can be factored through $\bigoplus_{t=j+1}^{n} X(t)$. But then, by Lemma 3.3, also f can be factored through $\bigoplus_{t=j+1}^{n} X(t)$, and thus $a \in e_i\pi_{ij}e_jA\varepsilon_{j+1}Ae_l$.

In all the three cases, we have shown that

$$e_iA\varepsilon_{j+1}Ae_l \cap e_iAe_jAe_l \subseteq e_i\pi_{ij}e_jA\varepsilon_{j+1}Ae_l.$$

It follows that $e_iA\varepsilon_{j+1}A \cap e_iAe_jA = e_i\pi_{ij}e_jA\varepsilon_{j+1}A$. $\square$

Remark 3.5. The direct consequence of the above lemma is the fact that $(A, \mathbf{e})$ is properly stratified. Moreover, the projective modules $P(i)$, $1 \leqslant i \leqslant n$, have the following property: if $[P(i) : \Delta(j)] \neq 0$, then $\underline{X}(j) \subseteq \underline{X}(i)$.

Remark 3.6. In view of Lemmas 3.1 and 3.4, the structure of proper standard modules and standard modules can be described explicitly. For every $i \in \Lambda$,

$$\overline{\mathcal{B}}(i) = \{ p_im_{ij}m_j \mid \underline{X}(i) \subseteq \underline{X}(j) \}$$

is a K-basis for the right proper standard module $\bar{\Delta}(i)$. Moreover,

$$\mathcal{B}(i) = \{ p_i\alpha_tm_{ij}m_j \mid \underline{X}(i) \subseteq \underline{X}(j) \}$$

is a K-basis of $\Delta(i)$ where $\{\alpha_t \mid 1 \leqslant t \leqslant d_i\}$ is a K-basis of

$$\left\{ \alpha \in \operatorname{End}_R\big(X(i)\big) \,\middle|\, \alpha \text{ cannot be factored through } \bigoplus_{t=i+1}^{n} X(t) \right\}.$$

In fact,

$$\bar{\Delta}(i) = e_iA/e_i \operatorname{rad} A\Bigg(\sum_{t\in\Lambda_i} e_t\Bigg)A \quad \text{and} \quad \Delta(i) = e_iA/e_iA\Bigg(\sum_{t\in\Lambda_i\setminus\{i\}} e_t\Bigg)A.$$

Another immediate consequence of the above facts is the multiplicity formula for the composition factors of the proper standard module:

$$\left[\bar{\Delta}(i):S(k)\right]=\begin{cases}1 & \text{if } \underline{X}(i)\subseteq \underline{X}(k),\\ 0 & \text{otherwise.}\end{cases}$$

4. Proof of necessity

Throughout this section, we assume that the endomorphism algebra $(A=\mathrm{End}_R(X),\mathbf{e})$ is a properly stratified algebra of Theorem 2.4. Note that the so-called Bernstein–Gelfand–Gelfand reciprocity relations reads as follows.

Proposition 4.1. *Let $(A,\mathbf{e})$ be a properly stratified algebra of Theorem* 2.4. *Then*

(i) $[P(i):\Delta(j)]=[\bar{\Delta}(j):S(i)]$ *for all* $1\leqslant i,j\leqslant n$;
(ii) $[P(i):\bar{\Delta}(j)]=[\Delta(j):S(i)]$ *for all* $1\leqslant i,j\leqslant n$.

Proof. The statement follows immediately from [1, Theorem 2.5] in combination with the existence of duality $\mathfrak{D}:\text{mod-}A\to\text{mod-}A$ that fixes the simple modules. □

The conditions (i) and (ii) of Theorem 2.4 will be proved in Lemmas 4.2 and 4.4.

Lemma 4.2. *Let $(A,\mathbf{e})$ be a properly stratified algebra of Theorem* 2.4. *Then*

$$\underline{X}(i)\cap\underline{X}(j)=\sum_{t\in\Lambda_i\cap\Lambda_j}\underline{X}(t)\quad \textit{for all } 1\leqslant i,j\leqslant n.$$

Proof. The statement will be proved by induction.

First, $\underline{X}(n)\subseteq\underline{X}(i)$ for every $1\leqslant i\leqslant n-1$. This follows from the fact that there is a nonzero homomorphism of $X(n)$ to every $X(i)$, and thus

$$\left[\bar{\Delta}(n):S(i)\right]\neq 0\quad \text{for all } 1\leqslant i\leqslant n-1.$$

Therefore, $[P(i):\Delta(n)]\neq 0$ for all $1\leqslant i\leqslant n-1$ by Proposition 4.1; this means that there is an epimorphism from $X(i)$ to $X(n)$. Thus, there is, by Proposition 2.1, a monomorphism from $X(n)$ to $X(i)$, as required.

Now, assume i is the largest index such that there is $j>i$ with $\underline{X}(j)\nsubseteq\underline{X}(i)$ and $\underline{X}(i)\cap\underline{X}(j)\neq\sum_{t\in\Lambda_i\cap\Lambda_j}\underline{X}(t)=\underline{Y}_{i,j}$; let j be the maximal with this property. Choose $\underline{x}\in\underline{X}(i)\cap\underline{X}(j)$ such that

$$\underline{x}+\underline{Y}_{i,j}\in\mathrm{soc}\left(\frac{\underline{X}(i)\cap\underline{X}(j)}{\underline{Y}_{i,j}}\right).$$

Define two homomorphisms $\underline{f}:\underline{X}(i)\to\underline{X}(j)$ and $\underline{g}:\underline{X}(j)\to\underline{X}(i)$ with $(\underline{x}_i)\underline{f}=\underline{x}$ and $(\underline{x}_j)\underline{g}=\underline{x}$. Correspondingly, since $\underline{x}\notin\underline{Y}_{i,j}$, f and g cannot be factored through

$\bigoplus_{t=j+1}^{n} X(t)$. Hence $[P(i) : \Delta(j)] \neq 0$ and $[\Delta(j) : S(i)] \neq 0$. Moreover, since A is properly stratified, $[\bar{\Delta}(j) : S(i)] \neq 0$. The homomorphism f defines a copy of $\bar{\Delta}(j)$ which appears in a $\bar{\Delta}$-filtration of $P(i)$. For any given element $r \in \operatorname{rad} R$, $rx \in \underline{Y}_{i,j}$. Since $\underline{X}(j) \nsubseteq \underline{X}(i)$, given any homomorphism $h : X(j) \to X(i)$, h cannot be a monomorphism and thus is induced by multiplication by an element in $\operatorname{rad} R$. Consequently, fh is factored through $\bigoplus_{t=j+1}^{n} X(t)$ which implies that $S(i)$ is not the composition factor in the copy of $\bar{\Delta}(j)$ induced by f. This fact contradicts $P(i) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\Delta})$. Thus, $\underline{X}(i) \cap \underline{X}(j) = \sum_{t \in \Lambda_i \cap \Lambda_j} \underline{X}(t)$ for all $1 \leqslant i, j \leqslant n$. The proof is completed. □

Lemma 4.3. *Let $(A, \mathbf{e})$ be a properly stratified algebra of Theorem* 2.4. *Then, for each $1 \leqslant i \leqslant n$,*

$$\big[\bar{\Delta}(i) : S(k)\big] \neq 0 \quad \text{if } \underline{X}(i) \subseteq \underline{X}(k); \qquad \big[\bar{\Delta}(i) : S(k)\big] = 0 \quad \text{otherwise}.$$

Proof. Let $k < i$. If $\underline{X}(i) \subseteq \underline{X}(k)$, then m_{ik} cannot be factored through $\bigoplus_{t=i+1}^{n} X(t)$. Consequently, $[\bar{\Delta}(i) : S(k)] \neq 0$. Since, every homomorphism from $X(i)$ to $X(k)$ which is not a monomorphism can be factored through an endomorphism mapping $X(i)$ into $\operatorname{rad} X(i)$, $[\bar{\Delta}(i) : S(k)] = 1$.

On the other hand, if $\underline{X}(i) \nsubseteq \underline{X}(k)$, then by Lemma 4.2, $\underline{X}(i) \cap \underline{X}(k) = \sum_{t \in \Lambda_i \cap \Lambda_k} \underline{X}(t)$ and thus every homomorphism from $X(i)$ to $X(k)$ can be factored through $\bigoplus_{t \in \Lambda_i \cap \Lambda_k} X(t)$. Consequently, in this case, $[\bar{\Delta}(i) : S(k)] = 0$. □

Lemma 4.4. *Let $(A, \mathbf{e})$ be a properly stratified algebra of Theorem* 2.4. *Then*

$$\underline{X}(j) \cap \left(\sum_{t=j+1}^{n} \underline{X}(t)\right) = \sum_{t=j+1}^{n} \big(\underline{X}(j) \cap \underline{X}(t)\big) \quad \text{for any } 1 \leqslant j \leqslant n.$$

Proof. We need to show

$$\underline{X}(j) \cap \left(\sum_{t=j+1}^{n} \underline{X}(t)\right) \subseteq \sum_{k \in \Lambda_j \setminus \{j\}} \underline{X}(k).$$

Indeed, the opposite inclusion is trivial, and by Lemma 4.2,

$$\sum_{t=j+1}^{n} \big(\underline{X}(j) \cap \underline{X}(t)\big) = \sum_{k \in \Lambda_j \setminus \{j\}} \underline{X}(k).$$

First, we are going to show that

$$e_1 \pi_{1j} e_j A \varepsilon_{j+1} A = e_1 \pi_{1j} e_j A \left(\sum_{t \in \Lambda_j \setminus \{j\}} e_t\right) A = e_1 A e_j A \cap e_1 A \varepsilon_{j+1} A;$$

recall that $\pi_{1j} = p_1 p_{ij} m_j$ of Lemma 3.4.

Since every homomorphism $h : X(1) \to X(j)$ can be written as a composition $p_{1j}h'$ with $h' : X(j) \to X(j)$, $e_1Ae_jA = e_1\pi_{1j}e_jA = e_1\pi_{1j}e_jA\varepsilon_jA \simeq e_jA$ and $e_1Ae_jA\varepsilon_{j+1}A = e_1\pi_{1j}e_jA\varepsilon_{j+1}A \simeq e_jA\varepsilon_{j+1}A$. Then,

$$\Delta(j) = e_jA/e_jA\varepsilon_{j+1}A \simeq e_1\pi_{1j}e_jA\varepsilon_jA/e_1\pi_{1j}e_jA\varepsilon_{j+1}A = e_1Ae_jA/e_1Ae_jA\varepsilon_{j+1}A.$$

Since $\underline{X}(j) \subseteq \underline{X}(1) = R$, we get that $[P(1) : \Delta(j)] = [\bar{\Delta}(j) : S(1)] = 1$ for all $1 \leqslant j \leqslant n$ by Lemma 4.3. It follows that

$$\Delta(j) \simeq e_1A\varepsilon_jA/e_1A\varepsilon_{j+1}A \simeq e_1Ae_jA/e_1Ae_jA \cap e_1A\varepsilon_{j+1}A.$$

Now $e_1Ae_jA\varepsilon_{j+1}A \subseteq e_1Ae_jA \cap e_1A\varepsilon_{j+1}A$ yields

$$e_1Ae_jA\varepsilon_{j+1}A = e_1\pi_{1j}e_jA\varepsilon_{j+1}A = e_1Ae_jA \cap e_1A\varepsilon_{j+1}A.$$

Moreover, by Proposition 4.1 and Lemma 4.3, $[P(j) : \Delta(k)] \neq 0$ only for $k \in \Lambda_j$, and therefore, $e_jA\varepsilon_{j+1}A = e_jA(\sum_{t\in\Lambda_j\setminus\{j\}} e_t)A$. Thus,

$$e_1\pi_{1j}e_jA\varepsilon_{j+1}A = e_1\pi_{1j}e_jA\left(\sum_{t\in\Lambda_j\setminus\{j\}} e_t\right)A = e_1Ae_jA \cap e_1A\varepsilon_{j+1}A.$$

Let $\underline{x} \in \underline{X}(j) \cap (\sum_{t=j+1}^{n} \underline{X}(t))$. Then there exist $r_l \in R$ for $j \leqslant l \leqslant n$ such that $\underline{x} = r_j\underline{x}_j = \sum_{t=j+1}^{n} r_t\underline{x}_t$. We can define two maps

$$X(1) \xrightarrow{p_{1j}} X(j) \xrightarrow{f} X(j) \xrightarrow{m_{j1}} X(1)$$

with $(\underline{x}_j)\underline{f} = r_j\underline{x}_j$ and

$$X(1) \xrightarrow{g} \bigoplus_{t=j+1}^{n} X(t) \xrightarrow{m} X(1)$$

with

$$g = (p_{1(j+1)}g_{j+1}, p_{1(j+2)}g_{j+2}, \ldots, p_{1n}g_n) \quad \text{and}$$
$$m = (m_{(j+1)1}, m_{(j+2)1}, \ldots, m_{n1})^{\mathrm{T}}$$

such that $(\underline{x}_t)\underline{g}_t = r_t\underline{x}_t$ for all $j+1 \leqslant t \leqslant n$. It follows that $\underline{gm} = \underline{p}_{1j}\underline{f}\underline{m}_{j1}$. For the corresponding homomorphism of local modules,

$$p_1p_{1j}fm_{j1}m_1 \in e_1Ae_jA \cap e_1A\varepsilon_{j+1}A = e_1\pi_{1j}e_jA\left(\sum_{t\in\Lambda_j\setminus\{j\}} e_t\right)A.$$

This implies that fm_{j1} is the sum of maps which can be factored through $\bigoplus_{t\in\Lambda_j\setminus\{j\}} X(t)$. Thus $\underline{x} \in \sum_{t\in\Lambda_j\setminus\{j\}} \underline{X}(t)$ and therefore

$$\underline{X}(j) \cap \left(\sum_{t=j+1}^{n} \underline{X}(t)\right) = \sum_{t\in\Lambda_j\setminus\{j\}} \underline{X}(t) = \sum_{t=j+1}^{n} \big(\underline{X}(j) \cap \underline{X}(t)\big). \qquad \square$$

5. Properties and examples

Properly stratified algebras constructed in the previous section enjoy some additional properties.

The following lemma establishes a relationship between our result and that of [5] on quasi-hereditary algebras.

Proposition 5.1. *Let $A = \mathrm{End}_R(X)$, with X as in Theorem* 2.4. *Let, moreover,* $\dim_K R = n$. *Then A is quasi-hereditary.*

Proof. Condition (ii) implies that the set $\{\underline{x}(i) \mid 1 \leqslant i \leqslant n\}$ is K-linearly independent. Thus, $\dim_K R \geqslant n$. Now

$$\dim_K R = \big[P(1) : S(1)\big] = \sum_{i=1}^{n} \big[P(1) : \Delta(i)\big]\big[\Delta(i) : S(1)\big]$$

and $[P(1) : \Delta(i)] = [\bar{\Delta}(i) : S(1)] = 1$ for all $1 \leqslant i \leqslant n$ by Lemma 4.3. If $\dim_k R = n$, $[\Delta(i) : S(1)] = 1$ for all $1 \leqslant i \leqslant n$. This implies $\Delta(i) = \bar{\Delta}(i)$. Hence $(A, \mathbf{e})$ is quasi-hereditary by Definition 2.2. $\square$

Proposition 5.2. *Let R, X, and A be as in Theorem* 2.4. *Considering X as a* (*right*) *A-module, we have $R \simeq \mathrm{End}_A(X_A)$.*

Proof. Define the map $\mu : R \to \mathrm{Hom}_A(X_A, X_A)^{\mathrm{opp}}$ by mapping r to $\mu_r : x \mapsto rx$. Clearly, μ is a monomorphism. In fact, we are going to show that μ is also an epimorphism. If $\rho \in \mathrm{End}_A(X_A, X_A)^{\mathrm{opp}}$, then $\rho(xe_i) = \rho(x)e_i$ (where e_i is the idempotent $p_i m_i$) and so $\rho(X(i)) \subseteq X(i)$. Hence $\rho_1 = \rho\,|_{X(1)}$ is an e_1Ae_1-endomorphism of $X(1)$ and $R \simeq \mathrm{End}_R(X(1), X(1)) \simeq e_1Ae_1$. Consequently, ρ_1 is of the form $r \mapsto sr$ for some $s \in R$. Now let $p_{1i} : X(1) \simeq R \to X(i)$ be the epimorphism corresponding to $\underline{p}_{1i}$ and let the element $\pi_{1i} \in A$ be defined as in the Lemma 3.4 for any $1 \leqslant i \leqslant n$. Then

$$\rho(x1) = \sum_{i=1}^{n} \rho(xe_i) = \sum_{i=1}^{n} \rho\big((x_i')\pi_{1i}\big) = \sum_{i=1}^{n} \big(\rho(x_i')\big)\pi_{1i} = \sum_{i=1}^{n} (sx_i')\pi_{1i} = sx. \qquad \square$$

Similarly as in Proposition 5.2, the following Propositions 5.3–5.5 all assume R, X, and A as in Theorem 2.4.

If the bimodule ${}_RX_A$ has the above property, then the number of nonisomorphic indecomposable summands of X_A is equal to the number of orthogonal primitive idempotents of R. Since R is a local algebra, we have the following proposition.

Proposition 5.3. *The module X_A is indecomposable. In fact, $X_A \simeq e_1A$ and $X_A \in \mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\Delta})$.*

Proof. Since $A = \mathrm{End}_R({}_RX, {}_RX)$, we have

$$e_1A_A \simeq \mathrm{Hom}_R({}_RR, {}_RX)_A \simeq X_A. \qquad \square$$

Proposition 5.4. *The indecomposable projective module $P(j)$ is isomorphic to a submodule of $P(1)$ for every $1 \leqslant j \leqslant n$.*

Proof. In fact, in proof of Lemma 3.4 we have shown this statement, since $e_1\pi_{1j}e_jA \simeq e_jA$. $\square$

Let $I(i)$ be the right indecomposable injective A-modules for all $1 \leqslant i \leqslant n$.

Proposition 5.5. *$P(1) \simeq I(1)$, i.e., $P(1)$ is projective–injective. Moreover, $P(1)$ is the unique indecomposable projective–injective module.*

Proof. Let $\xi \in P(1) = e_1A$, $\xi : X \xrightarrow{p_1} X(1) \xrightarrow{\xi_{1j}} X(j) \xrightarrow{m_j} X$. Since $\underline{X}(j) \subseteq \underline{X}(1) = R$ for all $1 \leqslant j \leqslant n$, we can write

$$\xi : X \xrightarrow{p_1} X(1) \xrightarrow{\xi_{1j}} X(j) \xrightarrow{m_{1j}} X(1) \xrightarrow{m_1} X.$$

Now consider the map $\zeta \in Ae_1$ given by

$$\zeta : X \xrightarrow{p_1} X(1) \xrightarrow{p_{1j}} X(j) \xrightarrow{D\xi_{1j}} X(1) \xrightarrow{m_1} X,$$

where $D = \mathrm{Hom}_K(\ , K)$. It is easy to see that $\xi \in \mathrm{Hom}_K(Ae_1, K) \simeq I(1)$ and since $D\zeta = \xi$, it follows that $P(1) \simeq I(1)$. In fact, from the above we can see that $e_1A = e_1Ae_1 = Ae_1$. For any other indecomposable projective module $P(j)$ ($j \neq 1$), we have $\mathrm{top}\, P(j) \simeq S(j)$. Since $P(j)$ is isomorphic to a submodule of $P(1)$ and $\mathrm{soc}\, P(1) \simeq \mathrm{soc}\, I(1)$, we have $\mathrm{soc}\, P(j) \simeq S(1)$. Since A is basic, $P(j)$ is not injective for any $j \neq 1$. $\square$

In what follows, we provide some illustrations of the main Theorem 2.4. On one hand, we describe in detail the case when $R = K[x]/\langle x^t \rangle$, on the other hand, we underline the differences to the situation when the endomorphism algebra A is quasi-hereditary. In addition, in Example 5.8 we point out the fact that there are other R-modules X than those described in Theorem 2.4 whose endomorphism algebras $A = \mathrm{End}_R(X)$ are properly stratified. To illustrate the regular representations of the constructed properly stratified algebra, we only show the filtrations of the projective module $P(1)$ (recall that every indecomposable projective module is isomorphic to a submodule of $P(1)$).

Example 5.6. Let $R = K[x]/\langle x^t\rangle$, $t \geqslant 1$. All the nonisomorphic local ideals are of the form $\underline{X}(i) = \langle \bar{x}^{i-1}\rangle$. Choosing any subset of local ideals which contains $\underline{X}(1) = R$, the endomorphism algebra of the corresponding direct sum of those local modules is properly stratified. Indeed, one can see immediately that conditions (i) and (ii) are satisfied. Moreover, all proper standard modules $\bar{\Delta}(i)$ are uniserial. In the general case, when we choose $X = \bigoplus_{i=1}^{m} X(t_i)$ with $m \leqslant t$ and $\underline{X}(t_1) = R$, it is immediate to see that

$$\bar{\Delta} = \left(1,\ \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array},\ \ldots,\ \begin{array}{c} m \\ \downarrow \\ m-1 \\ \vdots \\ 2 \\ \downarrow \\ 1 \end{array} \right)$$

and thus the choice of t_i's is only reflected in the structure of $\Delta(i)$'s. Indeed, $[\Delta(i) : S(i)] = \dim_K X(t_i) - \dim_K X(t_{i+1})$. In particular, if $t_i = i$, $1 \leqslant i \leqslant t$, $\bar{\Delta} = \Delta$, i.e. $(A, \mathbf{e})$ is quasi-hereditary.

Let us add that the endomorphism algebra $(A, \mathbf{e})$ of this example is quadratic if and only if either $(t_{j+1} - t_j) = 1$ for all $1 \leqslant j \leqslant m-1$, or $(t_{j+1} - t_j) = 2$ for all $1 \leqslant j \leqslant m-1$ and $(t - t_m) \leqslant 1$.

Example 5.7. Let $R = K[x, y]/\langle xy, x^3 - y^3\rangle$.

(a) Let $X = \bigoplus_{i+1}^{5} X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x}^2 + \bar{y}^2\rangle$, $\underline{X}(3) = \langle \bar{y}^2\rangle$, $\underline{X}(4) = \langle \bar{x}^2\rangle$, and $\underline{X}(5) = \langle \bar{x}^3\rangle$. In this case the condition (ii) is not satisfied. Thus $A = \mathrm{End}_R(X)$ is not properly stratified; in fact $[P(1) : S(1)] = 6$ while a Δ-filtration of $P(1)$ would require $[P(1) : S(1)] = 7$.

(b) Let $X = \bigoplus_{i=1}^{5} X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} + \bar{y}\rangle$, $\underline{X}(3) = \langle \bar{y}\rangle$, $\underline{X}(4) = \langle \bar{y}^2\rangle$, and $\underline{X}(5) = \langle \bar{x}^2\rangle$. In this case the condition (i) is not satisfied. Thus, again, A is not properly stratified.

(c) Let $X = \bigoplus_{i=1}^{5} X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} + \bar{y}\rangle$, $\underline{X}(3) = \langle \bar{y}\rangle$, $\underline{X}(4) = \langle \bar{y}^2\rangle$, and $\underline{X}(5) = \langle \bar{x}^3\rangle$. In this case both conditions (i) and (ii) are satisfied. Under the natural order, $(A, \mathbf{e})$ is properly stratified and the proper standard modules and the stratification of the projective module $P(1)$ can be described as follows:

$$\bar{\Delta} = \left(1,\ \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array},\ \begin{array}{c} 3 \\ \downarrow \\ 1 \end{array},\ \begin{tikzcd}[column sep=small, row sep=small] & 4 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & & 3 \arrow[dl] \\ & 1 & \end{tikzcd},\ \begin{tikzcd}[column sep=small, row sep=small] & 5 \arrow[d] & \\ & 4 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & & 3 \arrow[dl] \\ & 1 & \end{tikzcd} \right),$$

$$
P(1) = \begin{tikzcd}[column sep=tiny, row sep=small]
 & & & 1 & & & \\
 & & 3 \arrow[dl] & & 2 \arrow[dr] & & \\
 & 1 & & 4 \arrow[dlll] \arrow[dl] & & 1 & \\
2 \arrow[dr] & & 3 \arrow[dl] & & 5 \arrow[dl] & & 2 \arrow[dl] \\
 & 1 & & 4 \arrow[dl] \arrow[dr] & & 1 & \\
 & & 2 \arrow[dr] & & 3 \arrow[dl] & & \\
 & & & 1 & & &
\end{tikzcd} .
$$

(d) Let $X = \bigoplus_{i=1}^{6} X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} + \bar{y} \rangle$, $\underline{X}(3) = \langle \bar{y} \rangle$, $\underline{X}(4) = \langle \bar{x}^2 + \bar{y}^2 \rangle$, $\underline{X}(5) = \langle \bar{y}^2 \rangle$, and $\underline{X}(6) = \langle \bar{x}^3 \rangle$. Since $\dim_K R = 6$ and conditions (i) and (ii) are satisfied, A is quasi-hereditary:

$$
\Delta = \bar{\Delta} = \left(1 \ , \ \begin{tikzcd}[row sep=small] 2 \arrow[d] \\ 1 \end{tikzcd} \ , \ \begin{tikzcd}[row sep=small] 3 \arrow[d] \\ 1 \end{tikzcd} \ , \ \begin{tikzcd}[row sep=small] 4 \arrow[d] \\ 2 \arrow[d] \\ 1 \end{tikzcd} \ , \ \begin{tikzcd}[column sep=tiny, row sep=small] & 5 \arrow[dl] \arrow[dr] & \\ 3 \arrow[dr] & & 2 \arrow[dl] \\ & 1 & \end{tikzcd} \ , \ \begin{tikzcd}[column sep=tiny, row sep=small] & 6 \arrow[d] \arrow[dr] & \\ & 5 \arrow[dl] \arrow[dr] & 4 \arrow[d] \\ 3 \arrow[dr] & & 2 \arrow[dl] \\ & 1 & \end{tikzcd} \right)
$$

and the projective module $P(1)$ has the following filtration:

$$
P(1) = \begin{tikzcd}[column sep=tiny, row sep=small]
 & & 1 & & \\
 & 3 \arrow[dl] & & 2 \arrow[dr] & \\
1 & 5 \arrow[dl] \arrow[d] & & 4 \arrow[dr] & 1 \\
2 \arrow[d] & 3 \arrow[dl] & & 6 \arrow[dll] \arrow[d] & 2 \arrow[d] \\
1 & 4 \arrow[d] & & 5 \arrow[dll] \arrow[d] & 1 \\
 & 2 \arrow[dr] & & 3 \arrow[dl] & \\
 & & 1 & &
\end{tikzcd} .
$$

(e) To get a properly stratified algebra, it is not necessary to choose soc R as a direct factor of X. Let $X = \bigoplus_{i=1}^{n} X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} + \bar{y} \rangle$, $\underline{X}(3) = \langle \bar{y} \rangle$, $\underline{X}(4) = \langle \bar{y}^2 \rangle$. A is properly stratified:

$$
\bar{\Delta} = \left(1 \ , \ \begin{tikzcd}[row sep=small] 2 \arrow[d] \\ 1 \end{tikzcd} \ , \ \begin{tikzcd}[row sep=small] 3 \arrow[d] \\ 1 \end{tikzcd} \ , \ \begin{tikzcd}[column sep=tiny, row sep=small] & 4 \arrow[dl] \arrow[dr] & \\ 2 \arrow[dr] & & 3 \arrow[dl] \\ & 1 & \end{tikzcd} \right),
$$

and the stratification of the projective module $P(1)$ is as follows:

$$P(1) = \begin{tikzcd}[column sep=small, row sep=small]
 & & 1 & & \\
 & 3 \arrow[dl] & & 2 \arrow[dr] & \\
1 & & 4 \arrow[dll] \arrow[d] & & 1 \\
2 \arrow[d] & & 3 \arrow[dll] & & 2 \arrow[d] \\
1 & & 4 \arrow[dl] \arrow[dr] & & 1 \\
 & 2 \arrow[dr] & & 3 \arrow[dl] & \\
 & & 1 & &
\end{tikzcd}.$$

(f) The following example shows that $\Delta(i) \neq \bar{\Delta}(i)$ can happen for all $1 \leqslant i \leqslant n$. Let $X = \bigoplus_{i=1}^{3} X(i)$ with $\underline{X}(1) = R$, $\underline{X}(2) = \langle \bar{x} + \bar{y} \rangle$ and $\underline{X}(3) = \langle \bar{y}^2 \rangle$. The algebra A is properly stratified with

$$\Delta = \left(\begin{tikzcd}[row sep=small] 1 \arrow[d] \\ 1 \end{tikzcd} , \begin{tikzcd}[row sep=small] 2 \arrow[d] \\ 1 \arrow[d] \\ 2 \arrow[d] \\ 1 \end{tikzcd} , \begin{tikzcd}[column sep=small, row sep=small]
 & 3 \arrow[d] & \\
 & 2 \arrow[dl] \arrow[dr] & \\
1 \arrow[ddr] & & 3 \arrow[d] \\
 & & 2 \arrow[dl] \\
 & 1 &
\end{tikzcd} \right) \quad \text{and} \quad \bar{\Delta} = \left(1 , \begin{tikzcd}[row sep=small] 2 \arrow[d] \\ 1 \end{tikzcd} , \begin{tikzcd}[row sep=small] 3 \arrow[d] \\ 2 \arrow[d] \\ 1 \end{tikzcd} \right)$$

and the stratification of projective module $P(1)$ is as follows:

$$P(1) = \begin{tikzcd}[column sep=tiny, row sep=tiny]
 & 1 & & & \\
 & & 2 \arrow[ddrr] & & \\
1 & & & & \\
 & & 3 \arrow[dll] & & 1 \\
2 \arrow[dd] & & & & 2 \arrow[d] \\
 & & 3 \arrow[dd] & & 1 \\
1 & & & & \\
 & & 2 \arrow[dl] & & \\
 & 1 & & &
\end{tikzcd}.$$

Example 5.8. The following example illustrates a general situation when the endomorphism algebra is properly stratified. Consider the four-dimensional algebra $R = K[x, y]/\langle xy, x^2 - y^2 \rangle$. Let

$$X = R \oplus \left(R\bar{x} \oplus R/R\bar{x}^2 \right) \big/ \left\langle \bar{x}^2 - \left(\bar{y} + R\bar{x}^2 \right) \right\rangle \oplus R\bar{x}^2.$$

Thus, not all direct summands of X are local–colocal. It is easy to see that

$$\bar{\Delta} = \left(1 \;,\; \begin{array}{c} 2 \\ \downarrow \\ 1 \end{array} \;,\; \begin{array}{c} 3 \\ \downarrow \\ 2 \\ \downarrow \\ 2 \\ \downarrow \\ 1 \end{array} \right)$$

and that the right regular representation of $A = \mathrm{End}_R(X)$ allows the following filtrations:

$$\begin{tikzcd}[row sep=small, column sep=tiny]
& & & 1 & & & & & & 2 \arrow[ddrr] \\
& & & 2 \arrow[ddrr] & & & & & 2 \arrow[dr] \\
& & 2 \arrow[dr] & & & & & 3 \arrow[dr] & & 1 & & 1 & & 3 \arrow[dl] & & 3 \arrow[d] \\
A_A = & 3 \arrow[dr] & & 1 & & 1 & \oplus & & 2 \arrow[dr] & & & & 2 \arrow[dl] & & \oplus & 2 \arrow[d] & . \\
& & 2 \arrow[dr] & & & & & & & 2 \arrow[d] & & 2 \arrow[d] & & & & 2 \arrow[d] \\
& & & 2 \arrow[d] & & & & & & 1 & & 1 & & & & 1 \\
& & & 1
\end{tikzcd}$$

Acknowledgments

The authors are indebted to the referee for a number of helpful comments. Furthermore, they express their gratitude to Leonard Scott for pointing out the reference [7] to them, and for valuable suggestions resulting in improvements of several formulations.

References

[1] I. Ágoston, V. Dlab, E. Lukács, Stratified algebras, C. R. Math. Rep. Acad. Sci. Canada 20 (1998) 22–28.
[2] E. Cline, B.J. Parshall, L.L. Scott, Stratifying endomorphism algebras, Mem. Amer. Math. Soc. 124 (591) (1996).
[3] V. Dlab, Quasi-hereditary algebras revisited, An. Ştiinţ. Univ. Ovidius Constanţa Ser. Math. 4 (1996) 43–54.
[4] V. Dlab, Properly stratified algebras, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) 1–6.
[5] V. Dlab, P. Heath, F. Marko, Quasi-hereditary endomorphism algebras, Canad. Math. Bull. 38 (1995) 421–428.
[6] V. Futorny, S. König, V. Mazorchuk, Categories of induced modules and standardly stratified algebras, Algebr. Represent. Theory 5 (3) (2002) 259–276.
[7] V. Ginsburg, Perverse sheaves and $\mathbb{C}^*$-actions, J. Amer. Math. Soc. 4 (3) (1991) 483–490.
[8] P. Lakatos, On a theorem of V. Dlab, Algebr. Represent. Theory 3 (1) (2000) 99–103.
[9] W. Soergel, Kategorie $\mathcal{O}$, Preverse Garben Und Moduln Über Den Koinvarianten Zur Weylgruppe, J. Amer. Math. Soc. 3 (2) (1990) 421–445.